A method of determination of an acquisition program in order to maximize the total utility

Catalin Angelo IOAN

Abstract. This paper solves in a different way the problem of maximization of the total utility. The author uses the diofantic equations (equations in integers numbers) and after a decomposing in different cases, he obtains the maximal utility.

Keywords: utility, maximization, diophantic

1 A method of maximization the total utility

Let a consumer which has a budget of acquisition of two goods, in value of \(S \in \mathbb{N}\) u.m. The prices of the two goods \(x\) and \(y\) are \(p_x\) and \(p_y\) respectively. The marginal utilities corresponding to an arbitrary number of doses are in the following table:

<table>
<thead>
<tr>
<th>No. of dose</th>
<th>(U_{mx})</th>
<th>(U_{my})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(u_{11})</td>
<td>(u_{12})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(i)</td>
<td>(u_{i1})</td>
<td>(u_{i2})</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>(n)</td>
<td>(u_{n1})</td>
<td>(u_{n2})</td>
</tr>
</tbody>
</table>

We want in what follows to determine the number of doses \(x\), respectively \(y\) such that the total utility: \(U_t = \sum_{i=1}^{a} u_{ai} + \sum_{j=1}^{b} u_{aj}\) to be maximal.

Let therefore \(S_1 \leq S\) and the equation:

(1) \(ap_x + bp_y = S_1\)

Let denote with \(d=(p_x,p_y)\) the greatest common divisor of \(p_x\) and \(p_y\). We well know the fact that the equation has entire solutions it is necessary that \(d \mid S_1\). Also, we shall consider: \(S_1 > S - \min\{p_x,p_y\}\) because if \(S_1 \leq S - \min\{p_x,p_y\}\) with a supplementary unit of \(x\) or \(y\), the total utility will grow.

Dividing (1) at \(d\), we have:

(2) \(\frac{ap_x}{d} + \frac{bp_y}{d} = \frac{S_1}{d}\)

and with the notation: \(p'_x = \frac{p_x}{d}, p'_y = \frac{p_y}{d}\) follows:

(3) \(ap'_x + bp'_y = \frac{S_1}{d}\)

It is well known that for any relative prime numbers \(A,B \in \mathbb{N}\) it exist \(\alpha, \beta \in \mathbb{Z}\) (determined eventually with the Euclid algorithm) suc that: \(\alpha A + \beta B = 1\). Like \((p'_x,p'_y) = 1\) follows that \(\exists \alpha, \beta \in \mathbb{Z}\) such that:

(4) \(\alpha p'_x + \beta p'_y = 1, \alpha p_x + \beta p_y = d\)

We have therefore:

(5) \(\frac{ap'_x + bp'_y}{d} = \left(\alpha p'_x + \beta p'_y\right)\)

or, in other words:
(6) \( p', (a - \frac{S}{d} \alpha) = p', (\frac{S}{d} \beta - b) \).

Because \((p', p') = 1\) follows from (6) that it exist \( k \in \mathbb{Z} \) such that:

(7) \( a - \frac{S}{d} \alpha = kp' \gamma; \ \frac{S}{d} \beta - b = kp' \delta \)

or:

(8) \( a = kp' \gamma + \frac{S}{d} \alpha; \ b = -kp' \delta + \frac{S}{d} \beta \).

We can easily write (8) like:

(9) \( a = \frac{kp' \gamma + S \alpha}{d}; \ b = -\frac{kp' \delta + S \beta}{d} \).

We have, \( a, b \geq 0 \), and from (1): \( a \leq \frac{S}{p} x, b \leq \frac{S}{p} y \).

From (9) we have:

(10) \[
\begin{cases}
  k \geq -\frac{S \alpha}{p_y} \\
  k \leq \frac{S \beta}{p_x} \\
  k \leq \frac{S (d - \alpha p_x)}{p_x p_y} \\
  k \geq \frac{S (\beta p_y - d)}{p_x p_y}
\end{cases}
\]

From (4) and (10) follows:

(11) \[
\begin{cases}
  k \geq -\frac{S \alpha}{p_y} \\
  k \leq \frac{S \beta}{p_x} \\
  k \leq \frac{S \beta}{p_x} \\
  k \geq -\frac{S \alpha}{p_y}
\end{cases}
\]

therefore:

(12) \( k \in \left\lceil -\frac{S \alpha}{p_y}, \frac{S \beta}{p_x} \right\rceil \cap \mathbb{N} = S = \left\lceil \frac{-\alpha}{p_y}, \frac{-\beta}{p_x} \right\rceil \cap \mathbb{N} \).

The length of the interval: \( \left\lceil -\frac{S \alpha}{p_y}, \frac{S \beta}{p_x} \right\rceil \) is \( S \frac{\beta + S \alpha}{p_x p_y} = \frac{S (\beta p_y + \alpha p_x)}{p_x p_y} = \frac{S d}{p_x p_y} \). Because \( p_x p_y = d[p_x, p_y] \) (the least common multiple of the two numbers) we obtain that the length of the interval is: \( \frac{S}{[p_x, p_y]} \). Will be exist therefore \( \left\lceil \frac{S}{[p_x, p_y]} \right\rceil + 1 \) entire values of \( k \) (where with \([z]\) we have noted the entire part of \( z \)) who verify the acceptability conditions.

Let therefore: \( a_k = \frac{kp_y + S \alpha}{d}, b_k = -\frac{-kp_\delta + S \beta}{d} \) with \( k \) upper determined.

We have: \( a_{k+1} = \frac{(k + 1)p_y + S \alpha}{d} = a_k + \frac{p_y}{d} \) and \( b_{k+1} = -\frac{(k + 1)p_\delta + S \beta}{d} = b_k - \frac{p_\delta}{d} \).
Because: \( U_{t,k} = \sum_{i=1}^{a_k} u_{i1} + \sum_{j=1}^{b_k} u_{j2} \) and \( U_{t,k+1} = \sum_{i=1}^{a_k} u_{i1} + \sum_{j=1}^{b_k} u_{j2} = \sum_{i=1}^{a_k} u_{i1} + \sum_{i=a_k+1}^{b_k} u_{i1} - \sum_{j=1}^{b_k} u_{j2} \) where \( U_{t,k} \) is the total utility corresponding to \( k \).

If exist \( k \) such that: \( U_{t,k+1} < U_{t,k} \) then:

\[ \sum_{i=a_k+1}^{b_k} u_{i1} - \sum_{j=1}^{b_k} u_{j2} < 0 \text{ or other: } \sum_{i=a_k+1}^{b_k} u_{i1} < \sum_{j=1}^{b_k} u_{j2} \]

We have: \( \sum_{i=a_k+1}^{b_k} u_{i1} < \sum_{i=a_k+1}^{b_k} u_{i1} \) because the both terms of sum have \( \frac{P_x}{d} \) components, and the marginal utilities are a descending range, \( a_k \) being ascending and analogously: \( \sum_{j=1}^{b_k} u_{j2} > \sum_{j=1}^{b_k} u_{j2} \)

because the both terms of sum have \( \frac{P_y}{d} \) components, and the marginal utilities are a descending range, \( b_k \) being descending.

We have now:

\[ U_{t,k+2} = U_{t,k+1} + \sum_{i=a_k+1}^{b_k} u_{i1} - \sum_{j=1}^{b_k} u_{j2} < U_{t,k+1} + \sum_{i=a_k+1}^{b_k} u_{i1} - \sum_{j=1}^{b_k} u_{j2} = U_{t,k} + 2 \left( \sum_{i=a_k+1}^{b_k} u_{i1} - \sum_{j=1}^{b_k} u_{j2} \right) < U_{t,k} \]

Like a conclusion, the range of total utilities, once it reach a local maximum for a \( k \), it reach in that point a global maximum.

2 Example

<table>
<thead>
<tr>
<th>No. of dose</th>
<th>( U_{mx} )</th>
<th>( U_{my} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>14</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>13</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>7</td>
</tr>
</tbody>
</table>

\( p_x = 4, \ p_y = 6, \ S = 33 \).

Solution

We have \( \min\{p_x, p_y\} = 4 \), therefore \( S_1 \in (29, 33] \). Like \( (p_x, p_y) = 2 \) follows that \( S_1 \in \{30, 32\} \).

We have now: \( 4(-1)+6-1 = 2 \) therefore \( \alpha = -1 \) and \( \beta = 1 \). From (12), we obtain: \( k \in S_1 \left[ \frac{1}{6}, \frac{1}{4} \right] \cap N \).

Like a conclusion:

- \( S_1 = 30 \Rightarrow k \in \left[ \frac{30}{6}, \frac{30}{4} \right] \cap N = \{5, 6, 7\} \);
\[ S_1 = 32 \Rightarrow k \in \left[ \frac{32}{6}, \frac{32}{4} \right] \cap \mathbb{N} = \{6,7,8\}. \]

From the upper relations:

- \[ S_1 = 30 \Rightarrow a_k = \frac{6k - 30}{2} = 3k - 15, \quad b_k = \frac{-4k + 30}{2} = -2k + 15, \quad k \in \{5,6,7\}; \]
- \[ S_1 = 32 \Rightarrow a_k = \frac{6k - 32}{2} = 3k - 16, \quad b_k = -2k + 16, \quad k \in \{6,7,8\}. \]

It follows:

- \[ S_1 = 30 \Rightarrow \]
  - \( k = 5 \Rightarrow a_5 = 0, \quad b_5 = 5 \Rightarrow U_{1,5} = 20 + 16 + 15 + 14 + 13 = 78 \)
  - \( k = 6 \Rightarrow a_6 = 3, \quad b_6 = 3 \Rightarrow U_{1,6} = 10 + 8 + 7 + 20 + 16 + 15 = 76 \)
  - \( k = 7 \Rightarrow a_7 = 6, \quad b_7 = 1 \Rightarrow U_{1,7} = \text{non computing}! \)

- \[ S_1 = 32 \Rightarrow \]
  - \( k = 6 \Rightarrow a_6 = 2, \quad b_6 = 4 \Rightarrow U_{1,6} = 10 + 8 + 20 + 16 + 15 + 14 = 83 \)
  - \( k = 7 \Rightarrow a_7 = 5, \quad b_7 = 2 \Rightarrow U_{1,7} = 10 + 8 + 7 + 6 + 5 + 20 + 16 = 72 \)
  - \( k = 8 \Rightarrow a_8 = 8, \quad b_8 = 0 \Rightarrow U_{1,8} = \text{non computing}! \)

Finally, the maximal utility will be \( U_t = 83 \) for 2 goods x and 4 goods y.